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OPTIMAL DESIGN OF RIGID-PLASTIC SIMPLY SUPPORTED BEAMS UNDER DYNAMIC PRESSURE LOADING

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Abstract—Optimal design of a rigid-plastic stepped beam, loaded by a uniform pressure over a time interval $0 \le t \le t_1$, is discussed. Such beam dimensions are sought for which the beam of constant volume attains minimal central deflection. This optimization problem is solved by the method of mode form solutions. Exact solutions are also found. By numerical calculations a good accordance between the exact and approximate solutions is stated.

1. INTRODUCTION

The problem of optimal design of rigid-plastic stepped beams under impulsive loading was discussed by Lepik and Mróz[1, 2]. In these papers the mode form of motion was assumed. Exact solutions for this case were obtained by Lepik[3]. The case of pressure loading which is uniform over the beam and constant during the time interval $0 \le t \le t_1$ was examined in [1, 4]. In the paper[1] a solution for moderate loads was given (in this case we have only stationary plastic hinges in the beam). By increasing the load intensity moving plastic hinges appear and the solution of the problem becomes quite complicated. It was shown in the paper[4] that the mode form solution can also be applied in this case. In that paper the kinetic energies of the beam for instants $t = t_1 - 0$ and $t = t_1 + 0$ were equated. As it follows from the paper[3] better results can be obtained if we use the condition, suggested by Martin and Symonds[5].

The aim of this paper is to solve the optimization problem in question by the method of mode form solutions applying the condition by Martin and Symonds and to compare the results with the exact solutions. The basic equations are derived in Section 2. In Section 3 the loading stage is considered. Mode form and exact solutions are found in Sections 4 and 5. Numerical results are discussed in Section 6.

2. BASIC EQUATIONS

Let us consider a rigid-plastic beam of rectangular cross-section with piecewise constant thickness (Fig. 1(a)). The beam is simply supported at both edges. A uniform pressure load p^*, which is constant during the time interval $0 \le t \le t_1$ and is taken off at $t = t_1$, is applied.

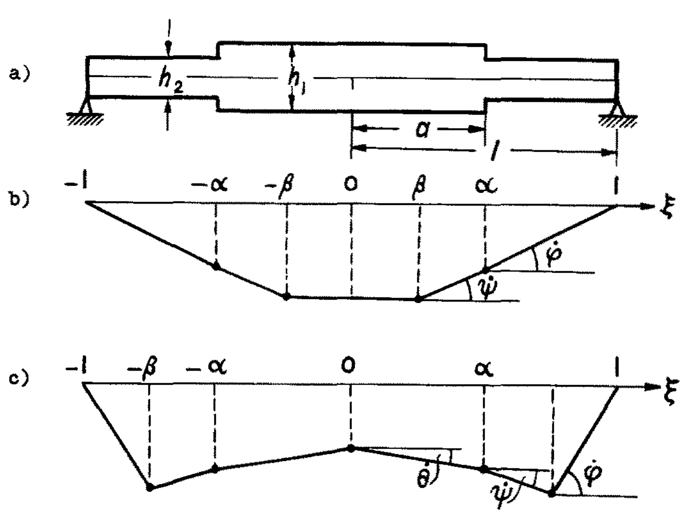

Fig. 1. (a) Beam dimensions; (b) yield mechanism for $\beta < \alpha$; (c) yield mechanism for $\beta > \alpha$.

Assuming constant volume of the beam, optimal parameters $\gamma = h_1/h_2 \geq 1$ and $\alpha = a/l$ for which the final deflection in the centre of the beam has minimum value, are to be found.

The equations of motion are

$$\frac{\partial M^*}{\partial x} = Q^*, \quad \frac{\partial Q^*}{\partial x} = -p^* + \rho B h^*(x)\frac{\partial^2 w^*}{\partial t^2}, \tag{1}$$

where M^* and Q^* are the bending moment and the shear force, ρ is the density, B, h^* and w^* respectively denote the width, height and deflection of the beam.

Now we shall introduce the following quantities

$$\xi = \frac{x}{l}, \quad \alpha = \frac{a}{l}, \quad \beta = \frac{b}{l}, \quad \tau = \frac{t}{t_1},$$

$$h = \frac{h^*}{h_2}, \quad N = \frac{3\sigma_0 V}{4\rho l^4 B}, \quad p = \frac{6p^*}{N\rho V}, \quad w = \frac{w^*}{Nlt_1^2},$$

$$M = \frac{4M^*}{\sigma_0 B h_2^2}, \quad Q^S = \frac{4lQ^*}{\sigma_0 B h_2^2}. \tag{2}$$

In these formulae σ_0 and V stand for the yield stress and volume of the beam. The latter quantity can be put into the form $V = 2Bh_2 l\Delta$, where $\Delta = \alpha\gamma + 1 - \alpha$. Since the volume V is specified we have

$$h_1 = \frac{V\gamma}{2Bl\Delta}, \quad h_2 = \frac{V}{2Bl\Delta}. \tag{3}$$

Making use of the formulae (2) and (3), eqns (1) acquire the following form (henceforth dots will denote differentiation with respect to the dimensionless time τ):

$$\frac{\partial M}{\partial \xi} = Q^s, \quad \frac{\partial Q^s}{\partial \xi} = -2\Delta^2 p + 6\Delta h(\xi)\ddot{w}. \tag{4}$$

In the present case we have two stages of loading: (1) the stage $0 \leq \tau \leq 1$, where the load acts, (2) the unloading stage $\tau > 1$ in which the motion proceeds by inertia. First we shall consider the loading stage $\tau < 1$. Since the load p has a constant value, a mode form solution will be valid for this stage. Let us take the deflection rate field in the form (Fig. 1b):

$$\dot{w} = \begin{cases} (1-\alpha)\dot{\varphi} + (\alpha - \beta)\dot{\psi} & \text{for } \xi \in [0, \beta] \\ (1-\alpha)\dot{\varphi} + (\alpha - \xi)\dot{\psi} & \text{for } \xi \in [\beta, \alpha] \\ (1-\xi)\dot{\varphi} & \text{for } \xi \in [\alpha, 1]. \end{cases} \tag{5}$$

Here $\dot{\alpha}$ and $\dot{\psi}$ stand for angular velocities, whose meaning follows from Fig. 1(b). In the cross-sections $\xi = \pm\alpha$ and $\xi = \pm\beta$ plastic hinges can appear; the value of β will be found in the following course of the solution. In the case of sufficiently high loads the formulae (5) become invalid. Now the hinges $\xi = \pm\beta$ appear in the thinner part of the beam and we have $\beta > \alpha$. Besides, by increasing the load p the bending moment in the middle of the beam decreases and can show there negative values. If in this region plastic hinges appear there the curves $\dot{w} = \dot{w}(\xi)$ must be concave since in the opposite case the work, dissipated at the hinges, would not be positive. Taking these facts into consideration we can postulate the following distribution for the deflection rates (Fig. 1c):

$$\dot{w} = \begin{cases} (1-\beta)\dot{\varphi} + (\beta - \alpha)\dot{\psi} + (\alpha - \xi)\dot{\theta} & \text{for } \xi \in [0, \alpha] \\ (1-\beta)\dot{\varphi} + (\beta - \xi)\dot{\psi} & \text{for } \xi \in [\alpha, \beta] \\ (1-\xi)\dot{\varphi} & \text{for } \xi \in [\beta, 1] \end{cases} \tag{6}$$

at which $\dot{\varphi} > 0$, $\dot{\psi} \leq 0$, $\dot{\theta} \leq 0$.

Since the bending moment cannot exceed its limit values, we have to fulfill the inequalities: $|M(\xi)| \leq \gamma^2$ for $\xi \in [0, \alpha]$ and $|M(\xi)| \leq 1$ for $\xi \in [\alpha, 1]$. Typical distribution of the bending moment $M(\xi)$ for the cases $\alpha < \beta$ and $\alpha > \beta$ are given in Fig. 2.

Now we shall integrate eqns (4). For this purpose we must differentiate the formulae (5) or (6) and insert the result into eqn (4). The integration constants must be calculated from the boundary conditions $Q^s(0) = M(1) = 0$ and from the continuity conditions for Q^s and M at $\xi = \alpha$ and $\xi = \beta$. Carrying out these calculations for the case $\beta > \alpha$ and denoting

$$q = \ddot{\varphi},\ R = \ddot{\psi},\ S = \ddot{\theta} \tag{7}$$

we get the following formulae

$$\begin{aligned} M(0) &= \Delta(A_1Q + B_1R) + \Delta^2 p \\ M(\beta) &= \Delta(A_2Q + B_2R) + \Delta^2 p(1 - \beta^2) \\ M(\alpha) &= \Delta(A_3Q + B_3R) + \Delta^2 p(1 - \alpha^2), \end{aligned}$$

where

$$\begin{aligned} A_1 &= -(1-\alpha)[2(1-\alpha)^2 + 3\alpha\gamma(2-\alpha)] \\ B_1 &= \gamma[\beta^2(3-\beta) - \alpha^2(3-\alpha)] \\ A_2 &= -(1-\alpha)\{2(1-\alpha)^2 + 3\gamma[\alpha(2-\alpha) - \beta^2]\} \\ B_2 &= \gamma[\beta^2(3-\beta) - \alpha^2(3-\alpha) + 3\beta^2(\alpha-\beta)] \\ A_3 &= -2(1-\alpha)^2(1-\alpha+3\alpha\gamma) \\ B_3 &= -3\gamma(1-\alpha)(\alpha^2-\beta^2). \end{aligned} \tag{9}$$

According to Fig. 1(b) it must be $M(0) \leq \gamma^2$, $M(\beta) \leq \gamma^2$, $M(\alpha) \leq 1$. In view of (8), these inequalities can be put into the form

$$A_1Q + B_1R \leq \frac{\gamma^2}{\Delta} - \Delta p \tag{10}$$

$$A_2Q + B_2R \leq \frac{\gamma^2}{\Delta} - \Delta p(1-\beta^2) \tag{10'}$$

$$A_3Q + B_3R \leq \frac{1}{\Delta} - \Delta p(1-\alpha^2). \tag{10''}$$

The case $\beta > \alpha$ can be treated in the analogous way. Here we have

$$Q = \frac{\Delta p}{2(1-\beta)} - \frac{1}{2\Delta(1-\beta)^3}. \tag{11}$$

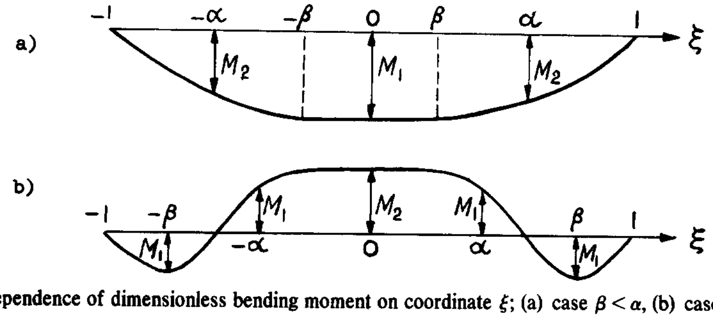

Fig. 2. Dependence of dimensionless bending moment on coordinate ξ; (a) case $\beta < \alpha$, (b) case $\beta > \alpha$.

The conditions $M(0) \geq -\gamma^2$, $M(\alpha) \geq -1$, $M(\beta) = 1$ can be put into the following form

$$A_4R + B_4S \geq \frac{C_4}{\Delta} + D_4\Delta p \tag{12}$$

$$A_5R \geq \frac{C_5}{\Delta} + D_5\Delta p \tag{12'}$$

$$A_6R + B_6S = \frac{C_6}{\Delta} + D_6\Delta p, \tag{12''}$$

where

$$\begin{aligned}
&A_4 = 2(1-\beta)^2(\beta-\alpha)[(\beta-\alpha)^2 - 3\alpha^2\gamma],\ B_4 = -4\alpha^3\gamma(1-\beta)^2 \\
&C_4 = -2\gamma^2(1-\beta)^2 - 2 + 4\beta + \beta^2 - 6\alpha\beta - 3(\gamma-1)\alpha^2 \\
&D_4 = -(1-\beta)^2[\beta(\beta-2\alpha) - 3(\gamma-1)\alpha^2] \\
&A_5 = 2(1-\beta)^2(\beta-\alpha)^3,\ C_5 = -4(1-\beta)^2 + 3(\beta-\alpha)^2 \\
&D_5 = -(1-\beta)^2(\beta-\alpha)^2,\ A_6 = (1-\beta)^2(\beta-\alpha)(2\alpha\gamma+\beta-\alpha) \\
&B_6 = \alpha^2\gamma(1-\beta)^2,\ C_6 = \alpha(\gamma-1)+\beta \\
&D_6 = -(1-\beta)^2[\alpha(\gamma-1)+\beta/3].
\end{aligned} \tag{13}$$

3. FIRST STAGE OF MOTION

Let us examine which modes of beam motion will be realized in the first stage $\tau < 1$. We shall begin with the condition $\beta < \alpha$; here we have the following five cases, for which the mode forms are shown in Fig. 3.

Case 1

Here a plastic hinge appears at the centre $\xi = 0$ and we have $\beta = 0$, $Q = R$. In the formula (10) the inequality sign must be replaced by an equality sign: since $C_1 = 0$, this equation takes the form

$$Q = R = \frac{\gamma^2 - p\Delta^2}{\Delta(A_1 + B_1)}. \tag{14}$$

The eqns (10) and (10′) coincide; the expression (10″) must be satisfied as a strong inequality.

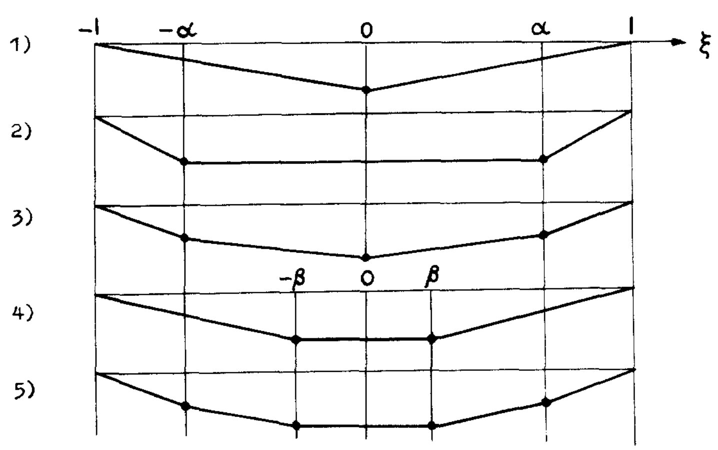

Fig. 3. Mode forms for $\beta < \alpha$.

Case 2

Now plastic hinges occur at $\xi = \pm\alpha$; consequently $\beta = 0$, $R = 0$. Replacing in (10″) the inequality by an equality we obtain

$$Q = \frac{1}{A_3}\left[\frac{1}{\Delta} - p\Delta(1-\alpha^2)\right]. \tag{15}$$

This case is valid when the inequality (10) is fulfilled.

Case 3

Here we have plastic hinges at the cross-sections $\xi = 0$ and $\xi = \pm\alpha$. Again $\beta = 0$ and expressions (10) and (10″) are satisfied as equalities; so we have two equations for calculating the values of Q and R: the inequalities $0 < |R| < |Q|$ must be satisfied.

For cases 1–3 the bending moment has the maximum at $\xi = 0$, consequently $\partial Q^s/\partial\xi \geq 0$ for $\xi = 0$. This requirement can be put into the form

$$p\Delta \geq 3\gamma[(1-\alpha)Q + \alpha R]. \tag{16}$$

Case 4

Here a stationary plastic hinge appears at $\xi = \pm\beta$. Now we have $M = \gamma^2$ in the central region $0 \leq |\xi| \leq \beta$. For this case $Q = R$, and the inequalities (10) and (10′) turn into a system of equations, from which the values of Q and β can be calculated. This case is realized, when $0 < \beta \leq \alpha$ and the inequality (10″) holds.

Case 5

Now let us assume plastic hinges appear at $\xi = \pm\beta$ and $\xi = \pm\alpha$. In this case all expressions (10)–(10″) must be satisfied as equalities. So we get a system for calculating the quantities Q, R and β. The inequalities $0 < \beta \leq \alpha$ and $0 < |R| < |Q|$ must hold.

Next we shall go over to the cases where $\beta > \alpha$; some mode forms for this case are shown in Fig. 4. For all these modes the quantity Q will be found from eqn (11).

Case 6

Here we have plastic hinges at $\xi = \pm\beta$; besides $R = S = 0$. The quantity β can be calculated from eqn (12″). The conditions (12) and (12′) must be satisfied as strong inequalities.

Case 7

In this case plastic hinges occur at $\xi = 0$ and $\xi = \pm\beta$. Now $R = S$, the inequality sign in (12)

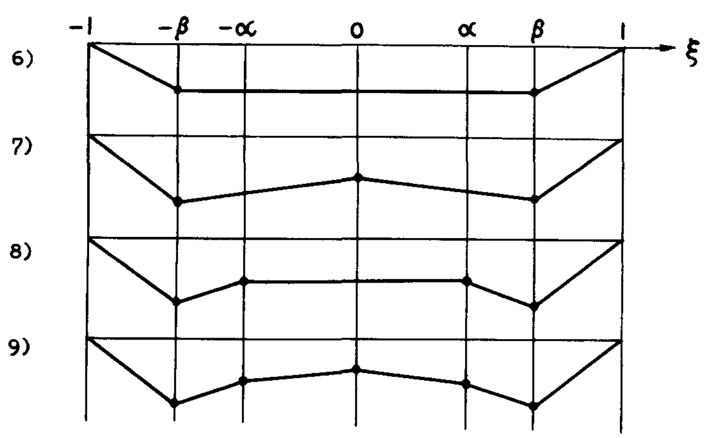

Fig. 4. Mode forms for $\beta > \alpha$.

has to be replaced by the sign of equality. The values of R and β can be found from (12) and (12″). It occurs when $R<0$ and (12′) is satisfied as a strong inequality.

Case 8

Here we have plastic hinges at $\xi=\pm\alpha$, $\xi=\pm\beta$. In this case $S=0$; the quantities R and β can be calculated from (12′) and (12″), assuming that the inequality sign in (12′) is again replaced by a sign of equality. The expression (12) has to be satisfied as a strong inequality, whereas it must be $R<0$.

Case 9

In this case we have plastic hinges at $\xi=0$, $\xi=\pm\alpha$, $\xi=\pm\beta$. Now all inequalities (12)–(12″) turn into equalities and we get a system from which the values of R, S and α can be calculated. The inequalities $R<S<0$ must hold good.

The cases 6–9 do not comprise all possible cases for $\beta>\alpha$. For very high loads more complicated modes appear and the number of plastic hinges can grow up to 8. Since such modes are practically not very relevant, they are not considered in this paper. Nevertheless a brief notice should be made of the limit case $p\to\infty$, $t_1\to 0$. Here the beam attains an impulse, which according to the theorem of momentum can be found from the formula $I=2Bl\rho h^*(x)v^*(x)$, where v^* denotes the initial velocity. In the case of a constant uniform pressure we have $h^*(x)v^*(x)=\text{const}$. This condition can be transformed into the dimensionless form $h_1v_1=h_2v_2$ and we get the initial velocity field presented in Fig. 5(b). So we can see that the homogeneous velocity field $v=\text{const}$ (Fig. 5a) does not correspond to the limit case $p\to\infty$, $t_1\to 0$. In spite of this the initial velocity distribution $v=\text{const}$ will hold good for some other problems (e.g. if the beam, moving with a constant velocity perpendicular to its axis, falls upon the supports).

At the end of this section we shall find the deflection rates and the central deflection for the instant $\tau=1$. Integrating eqns (7) we get $\dot\varphi=Q\tau$, $\dot\psi=R\tau$, $\dot\theta=S\tau$. These results will be inserted into eqns (5) and (6); taking $\tau=1$ we find the velocity field $\dot\omega(\xi,1)$. Since $\varphi=0.5Q\tau^2$, $\psi=0.5R\tau^2$, $\theta=0.5S\tau^2$ the central deflection at the end of the first stage is

$$w(0,1)=\begin{cases}0.5[(1-\alpha)Q+(\alpha-\beta)R] & \text{for } \beta<\alpha\\ 0.5[(1-\beta)Q+(\beta-\alpha)R+\alpha S] & \text{for } \beta>\alpha\end{cases}. \tag{17}$$

4. MODE FORM SOLUTION FOR THE SECOND STAGE

After removing the load p, also a mode form solution will be valid. Generally of course the motion in the first and second stages proceeds with different modes. Taking account of the complicated form of modes for high loads (Fig. 4) the following question may arise: is it well-founded in the second stage to confine ourselves only to the first modes; or do also higher modes take place? An answer to this question was given in [6]: it was shown that the second mode and likely also other higher modes are unstable and the motion always tends towards a first mode form solution. On the grounds of this fact we shall assume that in the stage $\tau>1$ only the modes 1–3 from Fig. 3 will be realized. All that was said about the cases 1–3 in Section 3 holds good also for $\tau>1$.

The mode form solution for $\tau>1$ is not an exact one, since the continuity condition for rate of deflections $\dot w$ is not fulfilled at $\tau=1$. Now we have to match the velocity fields at the end of

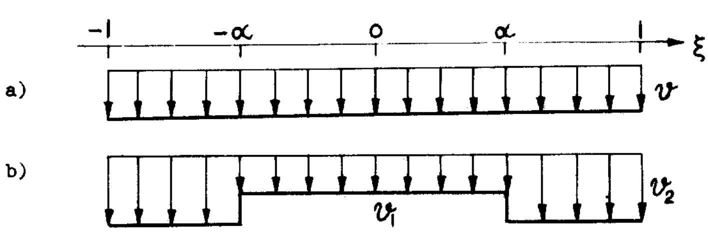

Fig. 5. Initial velocity fields in the case of impulsive loading.

the first stage and at the beginning of the second stage. In the paper [4] this problem was solved by equating the kinetic energies for both fields. As it follows from [3] more exact results can be achieved by the method of Martin and Symonds [5].

It was shown in [5] that the modal solution $\dot{u}_i(x, t) = T(t)\phi_i(x)$ approximates for given initial displacement rates $\dot{u}_i^0 = \dot{u}_i(x, 0)$ the real problem in the best way, when the following condition is fulfilled:

$$T(t_0)\int_{(V)} \rho\phi_i\phi_i \, dV = \int_{(V)} \rho\dot{u}_i^0\phi_i \, dV, \; i = 1, 2, 3. \tag{18}$$

For our problem we shall take $t_0 = 1$, $T(1) = 1$; $\dot{u}_i^0$ will be the deflection rate at the end of the first stage and ϕ_i the deflection rate at the beginning of the second state, i.e. $\dot{u}_i^0 = \dot{w}(\xi, 1-0)$, $\phi_i = \dot{w}(\xi, 1+0)$. Let us mark the values of φ, ψ, Q and R in the second stage with an asterisk. For mode form solutions the relation $\dot{\varphi}^* : \dot{\psi}^* = Q^* : R^*$ holds. Calculating the velocity fields from eqns (5) or (6) and inserting these results into (18) we obtain

$$\frac{\dot{\varphi}_*(1)}{Q_*} = \frac{L_1}{L_2}\frac{\dot{\varphi}(1)}{Q}. \tag{19}$$

The coefficients L_1 and L_2 will be calculated from the formulae:

$$L_1 = 2(1-\alpha)^2(3\alpha\gamma + 1 - \alpha)QQ_* + 3\alpha^2\gamma(1-\alpha)QR_* + 3\gamma(1-\alpha) \times(\alpha^2 - \beta^2)RQ_* + \gamma(\alpha-\beta)(2\alpha^2 + 2\alpha\beta - \beta^2)RR_* \text{ for } \beta < \alpha$$

$$L_1 = (1-\beta)[6\alpha\gamma(1-\alpha) + 2 + 2\beta - \beta^2 - 6\alpha + 3\alpha^2]QQ_* + 3\alpha^2\gamma \times(1-\beta)QR_* + (\beta-\alpha)[6\alpha\gamma(1-\alpha) + (\beta-\alpha)(3-\beta-2\alpha)]RQ_* + 3\alpha^2\gamma \times(\beta-\alpha)RR_* + 3\alpha^2\gamma(1-\alpha)SQ_* + 2\alpha^3\gamma SR_* \text{ for } \beta > \alpha,$$

$$L_2 = 2(1-\alpha)^2(3\alpha\gamma + 1 - \alpha)Q_*^2 + 6\alpha^2\gamma(1-\alpha)Q_*R_* + 2\alpha^3\gamma R_*^2.$$

Let us assume that the motion stops at the instant $\tau = \tau_f$. By integrating the equation $\ddot{\varphi}_* = Q_*$ we obtain

$$\dot{\varphi}(\tau_f) = \dot{\varphi}_*(1) + Q_*(\tau_f - 1) = 0$$

$$\varphi(\tau_f) = \varphi_*(1) + \dot{\varphi}_*(1)(\tau_f - 1) + \frac{1}{2}Q_*(\tau_f - 1)^2.$$

Eliminating the quantity τ_f from these equations and taking into account the continuity of deflections $\varphi(1) = \varphi_*(1)$, we get

$$\dot{\varphi}(\tau_f) = \varphi(1) - \frac{[\dot{\varphi}_*(1)]^2}{2Q_*}. \tag{20}$$

An analogous result also is valid for $\psi(\tau_f)$.

The residual central deflection will be found by integrating the second formula of (5); since $\beta = 0$, we obtain

$$w(0, t_f) = w(0, 1) + (1-\alpha)[\varphi(\tau_f) - \varphi(1)] + \alpha[\psi(\tau_f) - \psi(1)]$$

In view of (19) and (20) this result can be put into the form

$$w(0, t_f) = w(0, 1) - 0.5\left(\frac{L_1}{L_2}\right)^2[(1-\alpha)Q_* + \alpha R_*]. \tag{21}$$

We shall calculate the quantity $w(0, 1)$ in this formula according to eqns (17).

It is convenient to mark the regimes of motion by two numbers; they show which of the cases

1–9 participates in the loading and unloading stages. So number 72 (for example) tells us that for $P \neq 0$ the mode form 7 and for $P = 0$ the mode form 2 are realized.

5. EXACT SOLUTION

The results, obtained in Section 3 remain valid also for the loading stage in case of the exact solution. For the second stage we must construct a solution which guarantees the continuity of the deflection rates at $\tau = 1$. The deflection rate fields from Figs. 3 and 4 can also be used for $\tau > 1$, only now moving hinges at the cross-sections $\xi = \pm\beta$ may appear. In order to distinguish all possible regimes of motion, we shall use the following notation. Each regime is marked by a set of numbers; the first number shows the form of motion from Figs. 3 and 4 which takes place in the loading stage, the following numbers indicate the subsequence of motion forms for the second stage. The first number of the set also determines the number of the corresponding case (this numeration of cases is in accordance with Section 3). It follows from theoretical considerations and numerical calculations, that the following regimes of motion are possible.

Case 1

Here we have the mode form solution 11. The other possible regime 132 was not observed in carrying out the numerical calculations.

Case 2

In this case the motion in the second state $\tau > 1$ is more complicated. Let us introduce the functions

$$\begin{aligned} F(\alpha, \gamma) &= 3\alpha^2\gamma - 2(1-\alpha)(\gamma^2 - 1)(3\alpha\gamma + 1 - \alpha) \\ G(\alpha, \gamma) &= \alpha^2\gamma - (1-\alpha)(\gamma^2 - 1)[3\alpha\gamma + 2(1-\alpha)]. \end{aligned} \tag{22}$$

It was shown in paper [3] that three following subcases can be realized:

(i) if $F < 0$ the mode form solution 22 holds good;

(ii) if $F > 0$ and $g < 0$ we have the regime 231;

(iii) if $G > 0$ the regime 2541 takes place.

The equations of motion (4) were integrated for these subcases in paper [3].

Case 3

Here we have the regimes 331 and 332 (the latter did not occur within the considered ranges of α and γ).

Case 4

The regime 441 will be valid.

Case 5

Here the regimes 5531 and 5541 take place.

Case 6

In the unloading stage we have at first the motion form 6; it lasts until $\beta = \alpha$. The following motion proceeds according to case 2.

Cases 7 *and* 8

Here we have respectively motion forms 7 or 8. These go over to form 6 and the following motion proceeds as shown in the case 6.

Case 9

In this case the motion form 9 goes over to form 7; the motion further proceeds in conformity with case 7.

Now we have to integrate the equations of motion (4). For the motion forms 1–6 it can be done analytically (for details consult the paper [3], mentioned above). As to the forms 7–9, eqns (7)–(9) must be integrated numerically (e.g. by the method of Runge–Kutta). For the sake of conciseness the details of this work are not given here. For all regimes of motion the values of residual central deflection were calculated. Of course, the results are valid only if the limit values of the bending moments are not exceeded, in other words, the inequalities $|M| \le \gamma^2$ for $\xi \in (0, \alpha)$ and $|M| \le 1$ for $\xi \in (\alpha, 1)$ must be fulfilled. The validity of these inequalities was checked for each form of motion.

6. DISCUSSION

Let us first examine the case of a beam with uniform thickness, taking $\gamma = 1$. For dimensionless pressure $p \le 3$ the regime 11 is realized and we have $Q = R = 0.5(p-1)$. If $p > 3$ the regime 41 takes place and we obtain

$$\alpha = 1 - \sqrt{\left(\frac{3}{p}\right)}, \quad Q = R = \left(\frac{p}{3}\right)^{3/2}.$$

For both cases $Q_* = R_* = S_* = -0.5$. It follows from the formulae (17) and (21) that

$$w(0, \tau_f) = \begin{cases} \frac{p}{4}(p-1) & \text{for } p \le 3 \\ \frac{1}{12}[2p + 3(p-1)^2] & \text{for } p > 3. \end{cases} \tag{23}$$

The exact solution gives

$$w(0, \tau_f) = \begin{cases} \frac{p}{4}(p-1) & \text{for } p \le 3 \\ \frac{p}{18}(4p-3) & \text{for } p > 3. \end{cases} \tag{24}$$

If we take $p = 6$, the mode form solution has an error of 3.6%; increasing the load the error also increases; so for $p = 12$ it is 7.5% and in the limit case $p \to \infty$ we have 12.5%.

Now let us pass over to the general case $\gamma \ne 1$. To begin with, we fix the parameter α taking $\alpha = 0.8$. The beam's width B is specified, the quantity h_2 is changing with γ. Some calculations, which were carried out on computer for three values of p, are presented in Fig. 6. The symbol W in this figure denotes the ratio of central residual deflections for stepped and uniform beams. Above the horizontal axis sets of numbers characterizing the regimes of motion, which take place at the corresponding value of γ, are given. On the grounds of Fig. 6 the following consequences can be drawn.

(i) As was shown before [1, 3] in case of mode form solutions we have interval of non-uniqueness $\gamma_* \le \gamma \le \gamma^*$, where three modes coexist. In case of the exact solution the function $W = W(\gamma)$ is always unique.

(ii) In the case of the exact solution the beam motion always terminates in the mode forms 1 or 2 from Fig. 3. It is interesting to note that when the motion ends with form 2, the results obtained by the method of mode form solutions almost coincide with exact solutions, but in the opposite case, when motion stops with form 1 accordance between the two solutions is considerably worse. In order to clarify this fact we shall examine the role of each phase of motion on the residual deflections of the second stage $\tau > 1$.

If the motion stops with the mode form 2, then the main part of deflections is obtained namely during this phase. Taking for instance $p = 20$ and $\gamma = 1.8$ we have regime 662; the calculations show that 98.8% of the residual deflection in the second stage is attained in the phase 2. For the second example we choose $p = 50$, $\gamma = 1.8$; now the regime 7762 is realized. From the deflections in the second stage 2.2% are obtained in phase 7; 2.7% in phase 6 and 95.1% in the modal phase 2. On the grounds of these numerical examples it is clear why the mode form solution has such high exactness.

Now we shall consider the cases where the motion terminates in form 1. If the load is

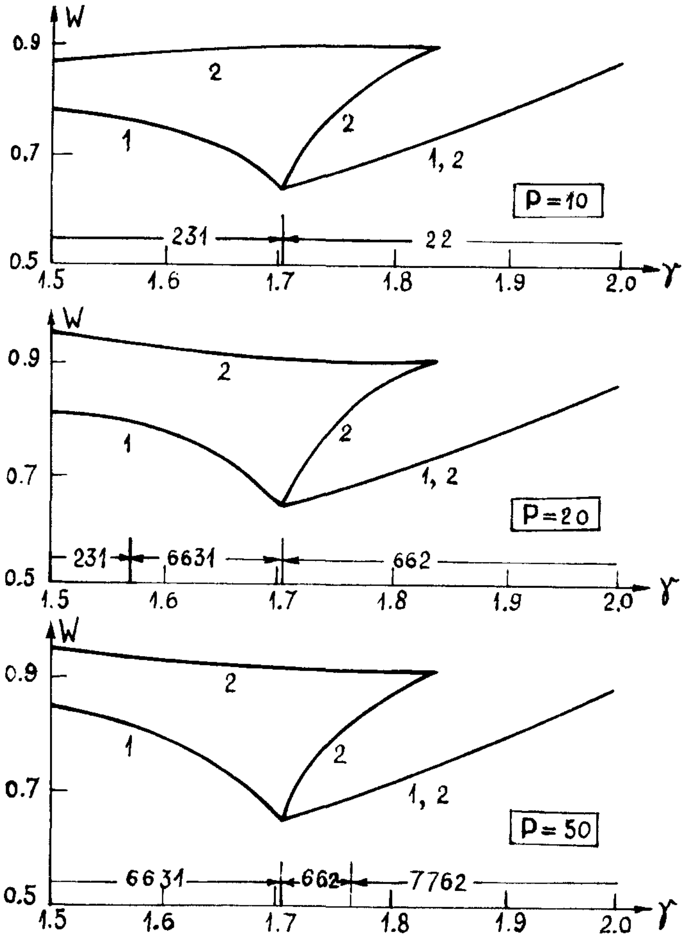

Fig. 6. Dependence of dimensionless deflections W on γ for $\alpha = 0.8$; 1, exact solution; 2, mode form solution.

sufficiently high, this mode is succeeded by the form of motion 3. This is not a mode, since the relation $\dot{\varphi} : \dot{\phi}$ is variable. It follows from the calculations that during this nonmodal form a considerable part of deflection is attained and therefore a mode form solution cannot be guarantee the necessary exactness. As an illustrative example, we shall take $p = 50$, $\gamma = 1.6$. Now we have the regime 6631; to the form 6 correspond 3.9% of the central deflection, to form 3, 70.1% and to the mode form 1, only 20.0%.

(iii) The curves 1 and 2 from Fig. 6 have minima at the same value of γ. This value can be calculated from the equation $F(\alpha, \gamma) = 0$, where the function $F(\alpha, \gamma)$ is defined according to the first formula of (22). It also follows from Fig. 6 that the minima of the function W in the case of the exact and mode form solutions differ insignificantly.

Now let us vary both parameters α and γ. Optimal values of these parameters for different loads are given in Table 1. (The limit case $p \to \infty$ corresponds to the initial velocity field given in

Table 1.

P	α	γ	W
2	0.74	1.51	0.49
3	0.75	1.54	0.53
5	0.76	1.56	0.59
10	0.76	1.58	0.62
20	0.77	1.62	0.64
50	0.79	1.68	0.65
∞	0.82	1.79	0.67

Fig. 5b.) It follows from the last column of the table that the reduction of deflection with respect to the beam of constant height is from 33 up to 51%. The quantities in Table 1 were calculated on the basis of the method of the mode form solutions, but it should be mentioned that they differ from the exact values trivially (e.g. for $p = 50$ the mode form solution gives $W = 0.654$ while in the case of an exact solution we have $W = 0.653$).

It was shown in paper [3] that when we move along the curve $F(\alpha, \gamma) = 0$, the value of W in the minium point alters very little. This conclusion is also valid for the problem discussed in this paper. Numerical calculations show that if we change the parameter γ in the interval $1.3 \leq \gamma \leq 2.4$ the values of W do not differ from their minimal values more than 5%. So there is no need to determine the exact optimal values for α and γ; it is enough to calculate the approximate optimal value for one parameter and find the other parameter from the equation $F(\alpha, \gamma) = 0$.

7. CONCLUSIONS

This paper completes a series of research work[1–4] about optimal design of rigid-plastic stepped simply supported beams. The cases of impulsive and dynamic pressure loading were considered. For integrating the equations of motion the method of mode form solutions was applied. In order to obtain error estimation for this method, also exact solutions were found. It was shown that the results which have been found by this method of mode form solutions coincide well with the exact solutions, when the velocity distribution has in the last phase of motion a trapezoidal form (for a triangular velocity distribution the accordance is much worse). Since in the case of an optimal solution the motion always terminates in a trapezoidal mode, the method of mode form solutions guarantees high exactness. Besides, it was shown that along the curve $F(\alpha, \gamma) = 0$ the deflections near the extreme point change insignificantly; this fact substantially enlarges the possibilities for optimal design of beams in question. So we can see that the method of mode form solutions in addition to its simplicity is fully reliable and it could be applied more widely than before for solving some complicated optimization problems.

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